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Stokes flow past a smooth cylinder

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SUMMARY

Ranger's solution of the two-dimensional Stokes flow past a smooth body is analyzed in detail. The body can be made convex or concave depending on the choice of its two parameters and concavity is found to be necessary but not sufficient for separation to occur. A relationship exists between the formation of a Stokesian wake and the curvature at the concave end of the body. Particular attention is given to the case when the body degenerates to a circular arc. The two-dimensional Stokes flow past a circular arc is strikingly similar to the axisymmetric Stokes flow past a spherical cap.

1. Introduction

Considerable interest has recently been stirred in Stokes flow because separation and reverse flow have been shown to occur even when fluid inertia is totally neglected. Dorrepaal, O'Neill and Ranger [1, 2] have examined Stokes flow past a spherical cap and found evidence of Stokesian wakes. The same authors along with Majumdar [3] have also detected separation and reverse flow in the form of axisymmetric Moffatt vortices in the Stokes flow past a closed torus. Michael and O'Neill [4] have looked at the Stokes flow about a spherical lens and again there is ample evidence of separation near the rim on the concave side of a thin concave-convex lens and on the flatter side of a thin asymmetric biconvex lens.

Now none of the above bodies have completely smooth surfaces. The cap and the lens have sharp-edged rims and the closed torus features an axisymmetric cusp. Although Stokes flows past such bodies exhibit separation, very little is known about the existence of wakes in Stokes flows past entirely smooth bodies. The sphere and the ellipsoid of revolution do not induce reverse flow but no work has yet been done on the Stokes flow past a smooth concave body of revolution. The solution of the governing equations for such a flow is an exceedingly difficult problem and yet progress has recently been made on an analogous two-dimensional problem. Ranger [5] has shown that an attached vortex is present within the concavity of a smooth concave cylinder in a two-dimensional Stokes flow. The purpose of Ranger's paper is to show how the solution is obtained using a new complex variable technique and other than establishing the existence of the vortex little analysis of the flow is done. In the present paper Ranger's solution is analyzed in detail and a number of interesting features discussed.

Finn and Noll [6] have shown that the fluid velocity in a two-dimensional Stokes flow past a finite fixed obstacle must necessarily become infinite at great distances from the obstacle. In other words it is not possible to have a uniform Stokes flow past a finite body in two dimensions. However Pearson and Proudman [7] have shown that the solution which becomes infinite most slowly at large distances from the body can be matched to a uniform stream at infinity. It seems reasonable then to consider this slowly diverging inner solution as the two-dimensional analogue to a uniform three-dimensional Stokes flow. The present paper gives strong evidence to support this assumption.

2. Statement of the problem

In two-dimensional flows the fluid velocity

$$Q = Ui + Vj \tag{2.1}$$

can be represented in terms of a stream function $\Psi(X, Y)$ as follows:

$$U = -\frac{\partial \Psi}{\partial Y}, \qquad V = \frac{\partial \Psi}{\partial X}. \tag{2.2}$$

In Stokes flow the stream function satisfies the biharmonic equation

$$\nabla^4 \Psi = 0 \tag{2.3}$$

and the fluid velocity Q must vanish on the boundary of the obstacle.

From [7] the condition on the stream function at infinity is

$$\Psi \sim \tfrac{1}{2} Y \log(X^2 + Y^2) \quad \text{as } X^2 + Y^2 \to \infty. \tag{2.4}$$

3. The solution

Ranger's solution is obtained using an inversion technique. The flow to be inverted is the two-dimensional Stokes flow past an oblate ellipse due to a stokeslet in the fluid along the extension of the ellipse's minor axis (Fig. 1).

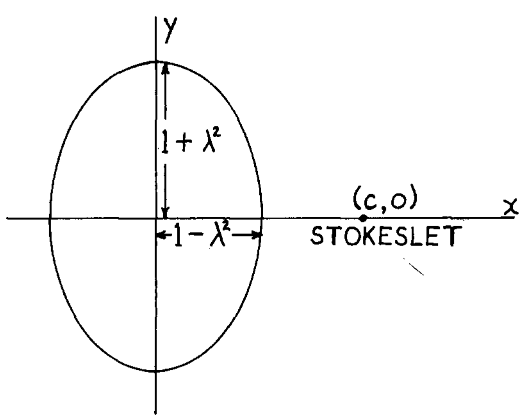

Figure 1. Relative positions of oblate ellipse and stokeslet ($1 - \lambda^2 < c$).

Parametric equations describing the boundary of the ellipse are

$$\left.\begin{aligned} x &= (1-\lambda^2)\cos\phi \\ y &= (1+\lambda^2)\sin\phi \end{aligned}\right\} \; 0 \leqslant \phi < 2\pi \tag{3.1}$$

where $0 \leqslant \lambda \leqslant 1$ is a parameter in the problem. These equations can be expressed in the form $\zeta = e^{i\phi}$ where

$$z = x + iy = \zeta - \lambda^2/\zeta. \tag{3.2}$$

The coordinates of the stokeslet in the x, y-plane are $(c, 0)$ where $c > 1 - \lambda^2$ is a second parameter. A stream function $\psi(x, y)$ can be defined from which the fluid velocity components

$$u = -\frac{\partial\psi}{\partial y} \qquad v = \frac{\partial\psi}{\partial x} \tag{3.3}$$

are derived. The stream function is biharmonic in (x, y) and the stokeslet condition is

$$\psi \sim -\tfrac{1}{2} y \log[(x-c)^2 + y^2] \quad \text{as} \quad (x-c)^2 + y^2 \to 0. \tag{3.4}$$

The solution to this problem is found in [5] and is stated here for reference:

$$\begin{aligned} u + iv = {} & \log \frac{2(\zeta + \bar\zeta) - d\zeta\bar\zeta - 1/d}{2(\zeta+\bar\zeta) - \zeta\bar\zeta/d - d} + \frac{A(1-\zeta\bar\zeta)}{d^2\zeta + \bar\zeta - d(1+\zeta\bar\zeta)} \\ & + \frac{(\zeta - \lambda^2/\zeta - 1/\bar\zeta + \lambda^2\bar\zeta)}{(1+\lambda^2/\bar\zeta^2)} [(\bar\zeta - d)^{-1} - (\bar\zeta - 1/d)^{-1} + A(d\bar\zeta - 1)^{-2}] \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} &\zeta(z) = \tfrac{1}{2} z + \tfrac{1}{2}(z^2 + 4\lambda^2)^{\frac{1}{2}} \qquad \text{(inverse of (3.2))} \\ &d = \zeta(c), \; A = \frac{(1-d^2)(1+\lambda^2)}{d(1+\lambda^2/d^2)} \end{aligned} \tag{3.6}$$

Now if this solution is inverted into the X, Y-plane using the transformation

$$X = \frac{x-c}{(x-c)^2 + y^2}, \qquad Y = \frac{y}{(x-c)^2 + y^2}, \tag{3.7}$$

then by the Inversion Theorem [8], the function

$$\Psi = \frac{\psi}{(x-c)^2 + y^2} \tag{3.8}$$

is biharmonic in (X, Y). In addition the stokeslet condition (3.4) becomes the condition at infinity given by (2.4). In other words by inverting the stokeslet problem in accordance with (3.7–8), we obtain the "uniform" two-dimensional Stokes flow past a cylinder whose cross-section is an inverted oblate ellipse. The velocity components in this new flow are given by

$$\begin{aligned} U &= -\frac{\partial\Psi}{\partial Y} = (u + 2Y\psi)\frac{X^2 - Y^2}{X^2 + Y^2} + (v - 2X\psi)\frac{2XY}{X^2 + Y^2}, \\ V &= \frac{\partial\Psi}{\partial X} = (u + 2Y\psi)\frac{2XY}{X^2 + Y^2} - (v - 2X\psi)\frac{X^2 - Y^2}{X^2 + Y^2}, \end{aligned} \tag{3.9}$$

where (u, v) are defined in (3.5).

4. The shape of the obstacle

Parametric equations describing the boundary of the obstacle in the X, Y-plane are obtained by substituting (3.1) into (3.7). The result is

$$\begin{aligned} X &= \frac{(1 - \lambda^2)\cos\phi - c}{(1 + \lambda^2)^2 + c^2 - 2c(1 - \lambda^2)\cos\phi - 4\lambda^2\cos^2\phi}, \\ Y &= \frac{(1 + \lambda^2)\sin\phi}{(1 + \lambda^2)^2 + c^2 - 2c(1 - \lambda^2)\cos\phi - 4\lambda^2\cos^2\phi}. \end{aligned} \tag{4.1}$$

When $\lambda = 0$, the obstacle is a circle centered at $(-c/(c^2 - 1), 0)$ with radius $1/(c^2 - 1)$. When $\lambda = 1$, the obstacle is a circular arc of radius $1/(2c)$ subtending an angle $2\alpha = 4\cot^{-1}(\frac{1}{2}c)$ at its center. For intermediate values of λ, the obstacle may be convex or concave depending upon the value of c. In Figure 2b the shape of the obstacle when $c = 1$, $\lambda^2 = \frac{1}{2}$ is shown. In all cases the body is symmetric about the X-axis.

Our first task is to determine when the cylinder is concave and when it is convex. One way to answer this question is to look for points on the boundary of the cylinder which have a

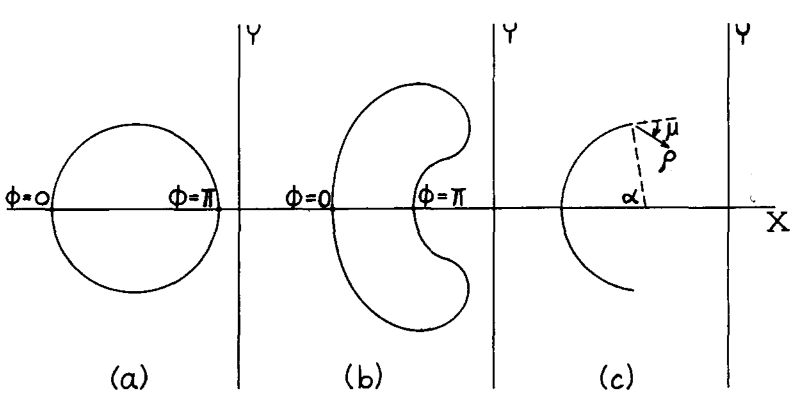

Figure 2. Various cylinder cross-sections.
(a) circle; $\lambda^2 = 0$. (b) kidney-shape; $c = 1$, $\lambda^2 = \frac{1}{2}$. (c) circular arc; $\lambda^2 = 1$.

vertical tangent. Clearly $\phi = 0, \pi$ are two such points. If there exists another point $\phi = \phi_0$ $(0 < \phi_0 < \pi)$ with this property the cylinder has a concave cross-section. If such a point does not exist the cylinder is convex.

Setting $dX/dt = 0$ in (4.1) where $t = \cos\phi$, we obtain

$$\cos\phi_0 = \frac{c}{1-\lambda^2} - \left[\frac{c^2}{(1-\lambda^2)^2} + \frac{(c+1+\lambda^2)(c-1-\lambda^2)}{4\lambda^2}\right]^{\frac{1}{2}}. \tag{4.2}$$

Now in order for $0 < \phi_0 < \pi$, we must have $-1 < \cos\phi_0 < 1$. From (4.2) this implies that

$$1 - \lambda^2 < c < \frac{1 + 6\lambda^2 + \lambda^4}{1-\lambda^2}. \tag{4.3}$$

This is the condition for the cylinder to be concave.

Expression (4.3) correctly predicts the presence or absence of concavity in the examples given in Figure 2. When $\lambda = 0$, (4.3) is inconsistent indicating the convexity of the circle for all values of c. When $\lambda = 1$, (4.3) is always true implying that circular arcs are always concave. When $c = 1$, $\lambda^2 = \frac{1}{2}$, the concavity condition is again satisfied.

5. Separation

Now that the geometry of the cylinder has been analyzed we turn next to the flow properties. In particular we look for cylinders which exhibit a Stokes wake. At the point of attachment of such a wake the fluid vorticity vanishes and so we must examine the vorticity on the boundary of the cylinder. From (3.7–8) the fluid vorticity is given by

$$\nabla^2\Psi = \frac{1}{X^2+Y^2}\nabla^2\psi + \frac{4uY}{X^2+Y^2} - \frac{4vX}{X^2+Y^2} + 4\psi. \tag{5.1}$$

Along the boundary $\zeta = e^{i\phi}$, the last three terms of (5.1) vanish and so separation occurs at those points ϕ for which

$$\nabla^2\psi|_{\zeta=e^{i\phi}} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)_{\zeta=e^{i\phi}} = 0. \tag{5.2}$$

Using (3.5) and omitting the algebraic details we find that the boundary vorticity vanishes at $\phi = \beta$ where

$$\cos\beta = -\frac{c(1-\lambda^2)}{4\lambda^2}. \tag{5.3}$$

Thus separation occurs as long as $\cos\beta > -1$ which implies

$$1 - \lambda^2 < c < \frac{4\lambda^2}{1-\lambda^2}. \tag{5.4}$$

This is the condition a cylinder must satisfy in order for a Stokes wake to form about its rear end $\phi = \pi$. When this condition is satisfied the point of separation is given by (5.3).

Now it is easy to see that the separation condition (5.4) implies the concavity condition (4.3). Thus any cylinder which exhibits a Stokes wake must necessarily be concave. On the other hand it is possible for a cylinder to be concave but exhibit no such wake. For the class of smooth bodies discussed in this paper therefore, concavity is necessary but not sufficient for the Stokes flow about the body to separate and form a wake.

In addition the separation condition does not hold for all values of λ. Condition (5.4) is only consistent when

$$3 - 2\sqrt{2} < \lambda^2 \leqslant 1. \tag{5.5}$$

When $\lambda^2 \leqslant 3 - 2\sqrt{2} = .1716$, the concavity of the obstacle (if it is concave at all) is not severe enough to induce separation.

A comparison of (4.2) with (5.3) reveals that

$$\cos \phi_0 \geqslant \cos \beta \tag{5.6}$$

with the equality occurring when $\lambda^2 = 1$ (circular arc). This means that the point of separation is always located along the concave surface of the cylinder (Fig. 3). In the case of the circular arc the dividing streamline emanates from the rim of the cylinder (Fig. 4b).

6. Geometric significance of separation condition

Because the cylinders being considered in this paper have smooth non-circular boundaries it is interesting to pause for a moment and examine whether boundary curvature might have an effect on the formation of Stokes wakes. Since these wakes always form within the concavity of the cylinder we begin by calculating the radius of curvature r_c at $\phi = \pi$. The result is

$$r_c|_{\phi=\pi} = \frac{(1+\lambda^2)^2}{[c + 1 - \lambda^2][1 + 6\lambda^2 + \lambda^4 - c(1-\lambda^2)]}. \tag{6.1}$$

Since the Cartesian coordinates of the point $\phi = \pi$ are $(-1/(c + 1 - \lambda^2), 0)$, the center of

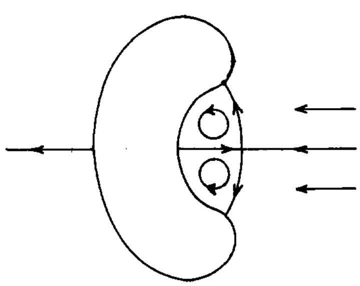

Figure 3. Stokes separation occurs along the cylinder's concave surface.

curvature at $\phi = \pi$ is $(C_c|_{\phi=\pi}, 0)$ where

$$C_c|_{\phi=\pi} = r_c|_{\phi=\pi} - \frac{1}{c+1-\lambda^2} = \frac{1}{c+1-\lambda^2}\left[\frac{(1+\lambda^2)^2}{1+6\lambda^2+\lambda^4-c(1-\lambda^2)} - 1\right]. \tag{6.2}$$

Interestingly enough the separation condition (5.4) is equivalent to the condition

$$C_c|_{\phi=\pi} \leqslant 0. \tag{6.3}$$

This statement is significant only for cylinders being considered in this paper. Another way of expressing (6.3) is

$$r_c|_{\phi=\pi} \leqslant \frac{1}{c+1-\lambda^2} \tag{6.4}$$

but no solely geometrical significance can be attached to the quantity on the right side of (6.4). This quantity does give the location of the rear end of the cylinder, but this means that it has positional significance rather than geometrical significance. Condition (6.4) nevertheless calls for a closer examination of the effects of boundary curvature on the formation of wakes in Stokes flow.

7. The circular arc ($\lambda^2 = 1$)

In Section 5 it was shown that the Stokes flow past a circular arc separates at the endpoints of the arc to form a Stokes wake along its concave side (Fig. 4b). This problem is the two-dimensional analogue of the uniform axisymmetric Stokes flow past a spherical cap and so a comparison of the two flows is in order. In Table 1 such a comparison is carried out. The cap and the arc are both assumed to have unit radius and the semi-angle α which both subtend at their respective centers provides the basis for comparison. The flow at infinity may be in either the positive or negative direction because Stokes flows are reversible.

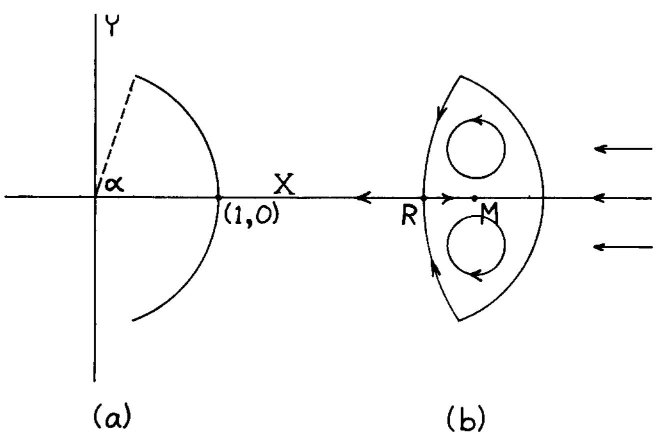

Figure 4. Relative positions of R, the rear stagnation point, and M, the point of maximum intra-wake velocity.

TABLE 1

α	Location of rear stagnation point		Location of intra-wake velocity maximum		Maximum intra-wake velocity	
	CAP	ARC	CAP	ARC	CAP	ARC
0°	1.0	1.0	1.0	1.0	0.0	0.0
15°	0.92	0.92	0.95	0.94	.0017	.0030
30°	0.72	0.72	0.82	0.81	.0079	.0142
45°	0.47	0.45	0.64	0.63	.0160	.0294
60°	0.18	0.16	0.44	0.43	.0236	.0437
75°	−0.10	−0.12	0.22	0.21	.0287	.0534
90°	−0.35	−0.37	0.0	0.0	.0305	.0569
105°	−0.57	−0.58	−0.22	−0.20	.0287	.0535
120°	−0.74	−0.75	−0.42	−0.40	.0238	.0443
135°	−0.86	−0.87	−0.60	−0.57	.0167	.0311
150°	−0.94	−0.95	−0.76	−0.73	.0090	.0168
165°	−0.99	−0.99	−0.89	−0.87	.0030	.0050
180°	−1.0	−1.0				

The similarities between the two flows are striking. The rear stagnation point and the intra-wake velocity maximum are almost identically positioned. The only noticeable difference between the two flows is the fact that the intra-wake velocities are greater in the case of the arc but this is to be expected since the two-dimensional problem has unbounded fluid velocity at infinity.

Another way of comparing the two flows is to examine how the dividing streamline leaves the rim in both cases. In the cap problem the angle of separation is known [1] to be

$$\mu_0 = 2\tan^{-1}(\tfrac{1}{3}\cot\tfrac{1}{2}\alpha) \tag{7.1}$$

where μ_0 is measured as shown in Figure 2c. This separation angle is found by expanding the stream function ψ_c near the rim of the cap and then using the first term of this expansion to calculate the angle μ_0 at which the dividing streamline $\psi_c = 0$ leaves the rim. Equivalently one can expand the velocity component $q_\mu = \partial\psi_c/\partial\rho$ near the rim and $\mu = \mu_0$ will define the ray emanating from the rim along which $q_\mu = 0$. We adopt the latter method to analyze the rim flow in the arc problem.

From Figure 2c it is easily seen that

$$q_\mu = U\cos(\alpha+\mu) - V\sin(\alpha+\mu), \tag{7.2}$$

$$\begin{aligned} X &= -\tfrac{1}{4}\sin\alpha + \rho\sin(\alpha+\mu),\\ Y &= \tfrac{1}{4}(1-\cos\alpha) + \rho\cos(\alpha+\mu). \end{aligned} \tag{7.3}$$

Now in order to calculate the leading term of the expansion for q_μ near the rim, we must have the corresponding expansions for U and V. These can be obtained from (3.9) if the expansions for u, v and ψ about the edge of the thin plate (oblate ellipse with $\lambda^2 = 1$) in the x, y-plane are known. But we know from Moffatt [9] that near the edge of the thin plate

$$\psi \sim \rho^{\frac{3}{2}}, \qquad u, v \sim \rho^{\frac{1}{2}}, \tag{7.4}$$

and so from (3.9) and (7.3) the leading terms (U_1, V_1) in the expansions for (U, V) are given by

$$U_1 = u_1 \cos\alpha - v_1 \sin\alpha, \quad V_1 = -u_1 \sin\alpha - v_1 \cos\alpha, \tag{7.5}$$

where (u_1, v_1) are the leading terms in the corresponding expansions for (u, v). Substituting (7.5) into (7.2) and simplifying yields the leading term $(q_\mu)_1$ in the expansion for q_μ:

$$(q_\mu)_1 = u_1 \cos\mu + v_1 \sin\mu. \tag{7.6}$$

The calculation of u_1 and v_1 from (3.5) is tedious but a very interesting result emerges when the work is finally completed:

$$(q_\mu)_1 = 4\rho^{\frac{1}{2}} \cot\tfrac{1}{2}\alpha \cos^2\tfrac{1}{2}\alpha \cos^3\tfrac{1}{2}\mu[1 - 3\tan\tfrac{1}{2}\alpha \tan\tfrac{1}{2}\mu]. \tag{7.7}$$

Setting $(q_\mu)_1$ equal to zero and solving for the separation angle μ_0 yields result (7.1). The angle at which the dividing streamline leaves the endpoint of the arc in two dimensions is exactly the same as the corresponding angle of separation in three dimensions regardless of the value of α.

This may or may not be surprising depending on one's point of view. The flow about the rim of a spherical cap is locally a two-dimensional flow and so one would not expect much difference between the angles of separation in the two problems. Nevertheless the fact that these two angles are exactly the same in all cases further highlights the striking similarity between the two flows. It seems likely therefore that the flow features in a uniform three-dimensional Stokes flow can be duplicated in two dimensions by requiring that the flow satisfy the logarithmic condition at infinity given in (2.4).

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